

Self-improvement of gradient estimate of heat flows on metric measure spaces

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Abstract

We prove that on a large family of metric measure spaces, if the ‘carré du champ’ Γ satisfies a p -gradient estimate of heat flows for some $p > 2$, then the metric measure space is $\text{RCD}(K, \infty)$. The argument relies on the non-smooth Bakry-Émery’s theory developed in [5, 15]. As an application, we provide another proof of the von Renesse-Sturm’s theorem on smooth metric measure space.

Keywords: Bakry-Émery theory, heat flow, gradient estimate, metric measure space

Contents

1	Introduction	1
2	Preliminaries	4
3	Main Results	6

1 Introduction

For any smooth connected Riemannian manifold M and any $K \in \mathbb{R}$, it is proved by von Renesse and Sturm in [14] that the following properties are equivalent

- 1) $\text{Ricci}_M \geq K$
- 2) there exists $p \in [1, \infty)$ such that for all $f \in C_c^\infty(M)$, all $x \in M$ and $t \geq 0$

$$|\text{DH}_t f|^p(x) \leq e^{-pKt} \text{H}_t |Df|^p(x). \quad (1.1)$$

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3) for all $f \in C_c^\infty(M)$, all $x \in M$ and $t \geq 0$

$$|DH_t f| \leq e^{-Kt} H_t |Df|, \quad (1.2)$$

where $H_t f$ is the solution to the heat equation with initial datum f .

In non-smooth setting, the notion of synthetic Ricci curvature bounds, or non-smooth curvature-dimension conditions, were proposed by Lott-Sturm-Villani (see [13] and [16] for $CD(K, \infty)$ and $CD(K, N)$ conditions) using optimal transport theory. Later on, by assuming the infinitesimally Hilbertianity (i.e. the Sobolev space $W^{1,2}$ is a Hilbert space), $RCD(K, \infty)$ condition which is a refinement of curvature-dimension condition, was proposed by Ambrosio-Gigli-Savaré (see [4] and [1]). It is known that $RCD(K, \infty)$ spaces are generalizations of Riemannian manifolds with lower Ricci curvature bound and their limit spaces, as well as Alexandrov spaces with lower curvature bound.

A natural question that arises concerns the relationship between Lott-Sturm-Villani's synthetic Ricci bound and Bakry-Émery's gradient estimate in the non-smooth setting. Let (X, d, \mathbf{m}) be a $RCD(K, \infty)$ space, it is proved (in [4]) that

$$|DH_t f|^2 \leq e^{-2Kt} H_t |Df|^2, \quad \mathbf{m} - \text{a.e.} \quad (1.3)$$

for any $f \in W^{1,2}$ and $t > 0$, where $H_t f$ is the heat flow from f and $|Df|$ is the minimal weak upper gradient (or weak gradient for simplicity) of f . In particular, by Hölder inequality we know

$$|DH_t f|^p \leq e^{-pKt} H_t |Df|^p, \quad \mathbf{m} - \text{a.e.} \quad (1.4)$$

for any $p \geq 2$. Furthermore, it is proved in [15] that the inequality (1.3) can be improved:

$$|DH_t f| \leq e^{-Kt} H_t |Df|, \quad \mathbf{m} - \text{a.e.} \quad (1.5)$$

In other words, inequality (1.4) holds for any $p \in [1, \infty]$.

Conversely, it is shown in [5] that the inequality (1.3) is sufficient to characterize $RCD(K, \infty)$ condition in the following way. Let (X, d, \mathbf{m}) be an infinitesimally Hilbertian space, we have a well-defined Dirichlet energy:

$$E(f) := \int |Df|^2 d\mathbf{m}$$

for any $f \in W^{1,2}(X, d, \mathbf{m})$. We denote the gradient flow of $E(\cdot)$ from f by $(H_t f)_t$. Assume further that the space (X, d, \mathbf{m}) has Sobolev-to-Lipschitz property, i.e. for any function $f \in W^{1,2}$ such that $|Df| \in L^\infty$, we can find a Lipschitz continuous function \tilde{f} such that $f = \tilde{f}$ \mathbf{m} -a.e. and $\text{Lip}(\tilde{f}) = \text{ess sup } |Df|$. If

$$|DH_t f|^2 \leq e^{-2Kt} H_t |Df|^2, \quad \mathbf{m} - \text{a.e.} \quad (1.6)$$

for any $f \in W^{1,2}$, and $t > 0$. Then (X, d, \mathbf{m}) is a $RCD(K, \infty)$ space.

The main goal of this paper is to prove that inequality (1.3) is sufficient to characterize the curvature-dimension condition of metric measure spaces. Equivalently, we prove a non-smooth version of von Renesse-Sturm's theorem.

Now, we introduce our main results in this paper and the ideas. Under Assumption 3.5 (i.e. the existence of a dense subspace \mathcal{A} of TestF such that $\Gamma(f) \in \mathbb{M}_\infty$ for any $f \in \mathcal{A}$) we prove in Theorem 3.6:

Theorem 1.1 (Improved Bakry-Émery theory). *Let (X, d, \mathbf{m}) be a metric measure space. If for any $f \in W^{1,2}$ we have the gradient estimate*

$$|\text{DH}_t f|^p \leq e^{-pKt} H_t |Df|^p, \quad \mathbf{m} - a.e. \quad (1.7)$$

for some $p \in [1, \infty)$. Then (X, d, \mathbf{m}) is a $\text{RCD}(K, \infty)$ space.

Since we do not have second order differentiation formula of relative entropy along Wasserstein geodesics, or Taylor's expansion in non-smooth setting. We can not simply use the argument in smooth metric measure space (see e.g. the proofs in [14]). The argument we use here is the so-called 'self-improved' method in Bakry-Émery's Γ -calculus, which was used in [15] to deal with the non-smooth problems.

It can be seen that the Assumption 3.5 is a very weak assumption. In the proof of 3.6 we know $\Gamma(f)^{\frac{p}{2}} \in \mathbb{M}_\infty$ for any $f \in \text{TestF}$. In order to find a dense subspace \mathcal{A} of TestF such that $\Gamma(f) \in \mathbb{M}_\infty$ for any $f \in \mathcal{A}$, we just need good cut-off functions, which is a weak local property of the space.

As a consequence, we prove in Theorem 3.7 the following local-to-global property.

Theorem 1.2 (Local-to-Global). *Let $M := (X, d, \mathbf{m})$ be a $\text{RCD}(K, \infty)$ metric measure space. Assume that X can be covered by (convex) open sets $\{\Omega_i\}_i$ such that $(\Omega_i, d, \mathbf{m})$ is $\text{RCD}(K_i, \infty)$ for some $K_i \in \mathbb{R}$, $i \in \mathbb{N}^+$. If for any $f \in W^{1,2}$ we have the gradient estimate*

$$|\text{DH}_t f|^p \leq e^{-pKt} H_t |Df|^p, \quad \mathbf{m} - a.e. \quad (1.8)$$

for some $p \in [1, \infty)$. Then (X, d, \mathbf{m}) is a $\text{RCD}(K, \infty)$ space.

From this theorem we prove Corollary 3.8, which is a well-known result on smooth metric measure space. Hence we obtain a new 'global' proof of this old result, without the use of Taylor's expansion method which is a 'local' approach.

At the end, as a direct consequence of Theorem 3.6 and Theorem 3.7, we obtain the following proposition. In [10] and [12], the authors provide a way to characterize lower Ricci bounds on RCD metric measure spaces using measure valued Ricci tensors. This proposition offers us an alternative approach to characterize the synthetic Ricci bound on RCD spaces.

Proposition 1.3 (Self-improvement of gradient estimate). *Let (X, d, \mathbf{m}) be a $\text{RCD}(K, \infty)$ metric measure space. If for any $f \in W^{1,2}$ we have the gradient estimate*

$$|\text{DH}_t f|^p \leq e^{-pK't} H_t |Df|^p, \quad \mathbf{m} - a.e. \quad (1.9)$$

for some $p \in [1, \infty)$ and $K' \in \mathbb{R}$. Then (X, d, \mathbf{m}) is a $\text{RCD}(K', \infty)$ space. In particular, we know

$$|\text{DH}_t f| \leq e^{-K't} H_t |Df|, \quad \mathbf{m} - a.e.. \quad (1.10)$$

The organization of this paper is the following. In Section 2 we introduce some basic notions and results on metric measure space and Sobolev space theory. Most of the results here can be found in [2, 4] and [15]. In Section 3 we prove our main theorem and its applications.

2 Preliminaries

In this paper, we start our discussion from a be a metric measure space (X, d, \mathbf{m}) . First of all, we need the following basic assumptions on (X, d, \mathbf{m}) , the notions and concepts in this assumption will be explained later.

Assumption 2.1. We assume that:

- (1) (X, d) is a complete, separable geodesic space,
- (2) $\text{supp } \mathbf{m} = X$, $\mathbf{m}(B_r(x)) < c_1 \exp(c_2 r^2)$ for every $r > 0$,
- (3) $W^{1,2}(X)$ is a Hilbert space,
- (4) (X, d, \mathbf{m}) has Sobolev-to-Lipschitz property

Remark 2.2. We can also use the language of Dirichlet form to study our problems, by assuming that the Dirichlet form is a so-called ‘Riemannian energy measure space’. It is known from [5] that this way is compatible with the current approach using Sobolev space on metric measure space.

The Sobolev space $W^{1,2}(X, d, \mathbf{m})$ is defined as in [2]. We say that $f \in L^2(X, \mathbf{m})$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions $(f_n) \subset L^2$, such that $f_n \rightarrow f$ and $\text{lip}(f_n) \rightarrow G$ in L^2 for some $G \in L^2(X, \mathbf{m})$, where $\text{lip}(f_n)$ is the local Lipschitz constant of f_n . It is known that there exists a minimal function G in \mathbf{m} -a.e. sense. We call the minimal G the minimal weak upper gradient (or weak gradient for simplicity) of the function f , and denote it by $|Df|$. It is known that the locality holds for $|Df|$, i.e. $|Df| = |Dg|$ a.e. on the set $\{f = g\}$. Furthermore, we have the lower semi-continuity: if $\{f_n\}_n \subset W^{1,2}(X, d, \mathbf{m})$ is a sequence converging to some f in \mathbf{m} -a.e. sense and such that $(|Df_n|)_n$ is bounded in $L^2(X, \mathbf{m})$, then $f \in W^{1,2}(X, d, \mathbf{m})$ and

$$\| |Df| \|_{L^2} \leq \liminf_{n \rightarrow \infty} \| |Df_n| \|_{L^2}.$$

We equip $W^{1,2}(X, d, \mathbf{m})$ with the norm

$$\|f\|_{W^{1,2}(X, d, \mathbf{m})}^2 := \|f\|_{L^2(X, \mathbf{m})}^2 + \| |Df| \|_{L^2(X, \mathbf{m})}^2.$$

We say that (X, d, \mathbf{m}) is an infinitesimally Hilbertian space if $W^{1,2}$ is a Hilbert space (see [4], [11] for more discussions).

On an infinitesimally Hilbertian space, we have a natural ‘carré du champ’ operator $\Gamma(\cdot, \cdot) : [W^{1,2}(M)]^2 \mapsto L^1(M)$ defined by

$$\Gamma(f, g) := \frac{1}{4} \left(|D(f+g)|^2 - |D(f-g)|^2 \right).$$

It can be seen that $\Gamma(\cdot, \cdot)$ is symmetric, bilinear and continuous. We also denote $\Gamma(f, f)$ by $\Gamma(f)$. We also have the following chain rule and Leibnitz rule (Lemma 4.7 and Proposition 4.17 in [1], see also Corollary 7.1.2 in [8],)

$$\Gamma(\Phi(f), g) = \Phi'(f)\Gamma(f, g) \quad \text{for every } f, g \in W^{1,2}, \quad \Phi \in \text{Lip}(\mathbb{R}), \Phi(0) = 0$$

and

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h) \quad \text{for every } f, g, h \in W^{1,2} \cap L^\infty.$$

We say that a metric measure space $M = (X, d, \mathbf{m})$ has Sobolev-to-Lipschitz property if: for any function $f \in W^{1,2}$ such that $|Df| \in L^\infty$, we can find a Lipschitz continuous function \tilde{f} such that $f = \tilde{f}$ \mathbf{m} -a.e. and $\text{Lip}(\tilde{f}) = \text{ess sup } |Df|$. In particular, by applying this property to the functions $\{d(z, \cdot) : z \in X\}$. We know the distance d is induced by Γ , i.e.

$$d(x, y) = \sup \{f(x) - f(y) : f \in W^{1,2} \cap C_b(X), |Df| \leq 1, \quad \mathbf{m} \text{ a.e.}\}.$$

Then we define the Dirichlet (energy) form $E : L^2 \mapsto [0, \infty]$ by

$$E(f) := \int \Gamma(f) d\mathbf{m}.$$

It is proved (see [2, 3]) Lipschitz functions are dense in energy in the sense that: for any $f \in W^{1,2}$ there is a sequence of Lipschitz functions $(f_n)_n \subset L^2(X, \mathbf{m})$ such that $f_n \rightarrow f$ and $\text{lip}(f_n) \rightarrow |Df|$ in L^2 . Moreover, if $W^{1,2}$ is Hilbert we know Lipschitz functions are dense (strongly) in $W^{1,2}$.

It can be proved that E is a strongly local, symmetric, quasi-regular Dirichlet form (see [2, 4, 5]). The Markov semigroup $(H_t)_{t \geq 0}$ generated by E is called the heat flow. For any $f \in L^2(X, \mathbf{m})$ we know $(0, \infty) \ni t \mapsto H_t f \in L^2 \cap D(\Delta)$ such that

$$\frac{d}{dt} H_t f = \Delta H_t f \quad \forall t \in (0, \infty),$$

and

$$\lim_{t \rightarrow 0} H_t f = f \quad \text{in } L^2.$$

Here the Laplacian is defined in the following way:

Definition 2.3 (Measure valued Laplacian, [10, 11, 15]). The space $D(\Delta) \subset W^{1,2}$ is the space of $f \in W^{1,2}$ such that there is a measure $\mu \in \text{Meas}(M)$ satisfying

$$\int \varphi d\mu = - \int \Gamma(\varphi, f) d\mathbf{m}, \quad \forall \varphi : M \mapsto \mathbb{R}, \quad \text{Lipschitz with bounded support.}$$

In this case the measure μ is unique and we denote it by Δf . If $\Delta f \ll m$, we denote its density with respect to \mathbf{m} by Δf .

We define $\text{TestF}(M) \subset W^{1,2}(M)$, the space of test functions as

$$\text{TestF}(M) := \left\{ f \in D(\Delta) \cap L^\infty : |Df| \in L^\infty \quad \text{and} \quad \Delta f \in W^{1,2}(M) \cap L^\infty(X, \mathbf{m}) \right\}.$$

It is known from [15] and [4] that $\text{TestF}(M)$ is an algebra and it is dense in $W^{1,2}(M)$ when M is a $\text{RCD}(K, \infty)$ metric measure space.

Lemma 2.4 (Lemma 3.2, [15]). *Let $M = (X, d, \mathbf{m})$ be a metric measure space satisfying Assumptions 2.1. Assume that the algebra generated by $\{f_1, \dots, f_n\} \subset \text{TestF}(M)$ is included in $\text{TestF}(M)$. Let $\Phi \in C^\infty(\mathbb{R}^n)$ be with $\Phi(0) = 0$. Put $\mathbf{f} = (f_1, \dots, f_n)$, then $\Phi(\mathbf{f}) \in \text{TestF}(M)$.*

Let $f \in \text{TestF}(M)$. We define the Hessian $H_f(\cdot, \cdot) : \{\text{TestF}(M)\}^2 \mapsto L^0(M)$ by

$$2\text{Hess}[f](g, h) = \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h)).$$

Then we have the following lemma.

Lemma 2.5 (Chain rules, [7], [15]). *Let $f_1, \dots, f_n \in \text{TestF}(M)$ and $\Phi \in C^\infty(\mathbb{R}^n)$ be with $\Phi(0) = 0$. Assume that the algebra generated by $\{f_1, \dots, f_n\} \subset \text{TestF}(M)$ is included in $\text{TestF}(M)$. Put $\mathbf{f} = (f_1, \dots, f_n)$, then*

$$|D\Phi(\mathbf{f})|^2 \mathbf{m} = \sum_{i,j=1}^n \Phi_i \Phi_j(\mathbf{f}) \langle \Gamma(f_i, f_j) \mathbf{m},$$

and

$$\Delta\Phi(\mathbf{f}) = \sum_{i=1}^n \Phi_i(\mathbf{f}) \Delta f_i + \sum_{i,j=1}^n \Phi_{i,j}(\mathbf{f}) \Gamma(f_i, f_j) \mathbf{m}.$$

The last lemma will be used in the proof of Theorem 3.6.

Lemma 2.6 (Lemma 3.3.6, [10]). *Let $\mu_i = \rho_i \mathbf{m} + \mu_i^s, i = 1, 2, 3$ be measures with $\mu_i^s \perp \mathbf{m}$. We assume that*

$$\lambda^2 \mu_1 + 2\lambda \mu_2 + \mu_3 \geq 0, \quad \forall \lambda \in \mathbb{R}.$$

Then we have

$$\mu_1^s \geq 0, \quad \mu_3^s \geq 0$$

and

$$|\rho_2|^2 \leq \rho_1 \rho_3, \quad \mathbf{m} - a.e..$$

3 Main Results

Firstly, we will discuss more about the measure valued Laplacian. Since E is quasi-regular, we know (see Remark 1.3.9(ii), [9]) that every function $f \in W^{1,2}$ has an quasi-continuous representative \bar{f} . And \bar{f} is unique up to quasi-everywhere equality, i.e. if \bar{f}' is another quasi-continuous representative, then $\bar{f}' = \bar{f}$ holds in a complement of an E -polar set. For more details, see Definition 2.1 in [15] and the references therein.

Definition 3.1. We define \mathbb{M}_∞ the space of $f \in D(\Delta) \cap L^\infty$ such that there exists a measure decomposition $\Delta f = \mu_+ - \mu_-$ with $\mu_\pm \in (W_+^{1,2})'$ where $W_+^{1,2} := \{\varphi \in W^{1,2} : \varphi \geq 0, \mathbf{m} - a.e.\}$, such that: every E -polar set is (Δf) -negligible and

$$\int \bar{\varphi} d(\Delta f) = - \int \Gamma(\varphi, f) d\mathbf{m}$$

for any $\varphi \in W^{1,2}$, the quasi-continuous representative $\bar{\varphi} \in L^1(X, \Delta f)$.

In this case, the measure $\bar{\varphi} \Delta f$ is well-defined.

In the next lemma we study the measure $\Delta\Gamma(f)^{\frac{p}{2}}$. Since $\Gamma(f)$ is not necessary a test function, and $\Phi(x) = x^{\frac{p}{2}}$ is not $C^2(\mathbb{R})$ when $p < 4$, we can not use Lemma 2.5 directly.

Lemma 3.2. *Let (X, d, \mathbf{m}) be a metric measure space satisfying assumptions 2.1. Let $f \in \text{TestF}$ such that $\Gamma(f), \Gamma(f)^{\frac{p}{2}} \in \mathbb{M}_\infty$, $p \geq 2$ and*

$$\frac{1}{p}\Delta\Gamma(f)^{\frac{p}{2}} - \Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f)d\mathbf{m} \geq K\Gamma(f)^{\frac{p}{2}}d\mathbf{m}. \quad (3.1)$$

Then we have:

$$\frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\Delta_{ac}\Gamma(f) + \frac{1}{2}\left(\frac{p}{2}-1\right)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))d\mathbf{m} \geq \left(\Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f) + K\Gamma(f)^{\frac{p}{2}}\right)d\mathbf{m} \quad (3.2)$$

as measures, where $\Delta_{ac}\Gamma(f)$ is the absolutely continuous part in the measure decomposition $\Delta\Gamma(f) = \Delta_{ac}\Gamma(f) + \Delta_{sing}\Gamma(f)$ with respect to \mathbf{m} .

Proof. We have the decomposition of the measure $\frac{1}{p}\Delta\Gamma(f)^{\frac{p}{2}}$ with respect to \mathbf{m} :

$$\frac{1}{p}\Delta\Gamma(f)^{\frac{p}{2}} = \frac{1}{p}\Delta_{sing}\Gamma(f)^{\frac{p}{2}} + \frac{1}{p}\Delta_{ac}\Gamma(f)^{\frac{p}{2}}.$$

From (3.1) we know the singular part $\frac{1}{p}\Delta_{sing}\Gamma(f)^{\frac{p}{2}}$ of the measure $\frac{1}{p}\Delta\Gamma(f)^{\frac{p}{2}}$ is non-negative.

Let $\overline{\Gamma(f)}$ be the quasi-continuous representation of $\Gamma(f)$, we assert that

$$\frac{1}{p}\Delta\Gamma(f)^{\frac{p}{2}} \leq \frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\Delta\Gamma(f) + \frac{1}{2}\left(\frac{p}{2}-1\right)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))d\mathbf{m} \quad (3.3)$$

as measures, where the right hand side of (3.3) is understood as $+\infty$ on $\{\Gamma(f) = 0\}$.

From assumptions we know $\Gamma(f), \Gamma(f)^{\frac{p}{2}} \in D(\Delta)$, by chain rule we know

$$\int \varphi d\Delta\Gamma(f)^{\frac{p}{2}} = - \int \Gamma(\varphi, \Gamma(f)^{\frac{p}{2}}) d\mathbf{m} = - \int \frac{p}{2}\Gamma(f)^{\frac{p}{2}-1}\Gamma(\varphi, \Gamma(f)) d\mathbf{m}$$

for any Lipschitz function φ with bounded support.

Then, by Leibniz rule and chain rule we know $\varphi\Gamma(f)^{\frac{p}{2}-1} \in W_{loc}^{1,2}(\{\Gamma(f) > 0\})$, according to Definition 3.1 we have

$$\begin{aligned} - \int \varphi \overline{\Gamma(f)}^{\frac{p}{2}-1} \Delta\Gamma(f) d\mathbf{m} &= \int \Gamma(\varphi \Gamma(f)^{\frac{p}{2}-1}, \Gamma(f)) d\mathbf{m} \\ &\leq \int \varphi \left(\frac{p}{2}-1\right) \Gamma(f)^{\frac{p}{2}-2} \Gamma(\Gamma(f)) d\mathbf{m} + \int \Gamma(f)^{\frac{p}{2}-1} \Gamma(\varphi, \Gamma(f)) d\mathbf{m}. \end{aligned}$$

Therefore, we know (3.3) is true. In particular, we know

$$\begin{aligned} \frac{1}{p}\Delta_{ac}\Gamma(f)^{\frac{p}{2}} &\leq \frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\Delta_{ac}\Gamma(f) + \frac{1}{2}\left(\frac{p}{2}-1\right)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))d\mathbf{m} \\ &= \frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\Delta_{ac}\Gamma(f) + \frac{1}{2}\left(\frac{p}{2}-1\right)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))d\mathbf{m} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \overline{\Gamma(f)}^{\frac{p}{2}-1} \Delta_{\text{sing}} \Gamma(f) &\geq \frac{1}{p} \Delta_{\text{sing}} \Gamma(f)^{\frac{p}{2}} \\ &\geq 0. \end{aligned}$$

In conclusion, combining with (3.3), the inequality (3.1) becomes (3.2). \square

The following lemma will be used in the proof in Theorem 3.6.

Lemma 3.3. *Let $P(r) : [0, \infty) \mapsto [-\frac{1}{4}, \infty)$ be a function defined as*

$$P(r) = r - \frac{1}{4(r+1)}.$$

Let $a_0 \geq 0$ be an arbitrary initial datum, we define a_n recursively by the formula

$$a_{n+1} = P(a_n).$$

Then there exists a integer N_0 such that $0 \leq a_{N_0} < 1$ and $-\frac{1}{4} \leq a_{N_0+1} < 0$.

Conversely, for any $a \in [0, 1)$ and $b > a$, there exists a sequence a_0, \dots, a_{N_0} defined by the recursive function P such that $a_0 > b$ and $a_{N_0} = a$.

Proof. It can be seen that $a_{n+1} < a_n$. If $a_0 \geq 0$, by monotonicity we know $a_n - a_{n+1} \in [\frac{1}{4(a_0+1)}, \frac{1}{4}]$ for any $n \in \mathbb{N}$. So there must exists a unique N_0 such that $0 \leq a_{N_0} < 1$ and $-\frac{1}{4} \leq a_{N_0+1} < 0$. Conversely, since $P(r)$ is strictly monotone on $[0, \infty)$, we know $P^{-1}(r) : [-\frac{1}{4}, \infty) \mapsto [0, \infty)$ is well defined. And $(P^{-1})^{(n+1)}(a) - (P^{-1})^{(n)}(a) \in [\frac{1}{4((P^{-1})^{(n+1)}(a)+1)}, \frac{1}{4}]$ for any $n \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that $(P^{-1})^{(N_0)}(a) \geq b$. Therefore, we can pick $a_0 = (P^{-1})^{(N_0)}(a)$, and $a_{N_0} = (P)^{(N_0)}(a_0) = a$ fulfils our request. \square

Lemma 3.4 (Density of test functions in $W^{1,2}(X, d, \mathbf{m})$, see Remark 2.5, [5]). *Let (X, d, \mathbf{m}) be a metric measure space satisfying Assumption 2.1. Assume that for any $f \in W^{1,2}$ we have the p -gradient estimate*

$$|DH_t f|^p \leq e^{-pKt} H_t |Df|^p, \quad a.e. \quad (3.4)$$

for some $p \in [1, \infty)$. Then the space of test functions $\text{TestF}(X, d, \mathbf{m})$ is dense in $W^{1,2}$.

Proof. As we discussed in the preliminary section, the space

$$\mathbb{V}^1 := \left\{ \varphi \in W^{1,2} : \Gamma(\varphi) \in L^\infty(X, \mathbf{m}) \right\}$$

is dense in $W^{1,2}$. We also know that the

$$\mathbb{V}_\infty^1 := \left\{ \varphi \in W^{1,2} \cap L^\infty : \Gamma(\varphi) \in L^\infty(X, \mathbf{m}) \right\}$$

in dense in L^2 , and \mathbb{V}_∞^1 is invariant under the action $(H_t)_t$ by (3.4). Hence by a approximation argument (see e.g. Lemma 4.9 in [4]), we know \mathbb{V}_∞^1 is dense in $W^{1,2}$. Similarly, by a semigroup mollification (see e.g. page 351, [5]) we can prove that

$$\mathbb{V}_\infty^2 := \left\{ \varphi \in \mathbb{V}_\infty^1 : \Delta\varphi \in W^{1,2} \cap L^\infty(X, \mathbf{m}) \right\}$$

is dense in $W^{1,2}$. □

The following assumption is basic and necessary in Bakry-Émery theory.

Assumption 3.5. [Property: existence of good algebra] We assume the existence of a dense subspace \mathcal{A} in $\text{TestF}(X, d, \mathbf{m})$ w.r.t $W^{1,2}$ norm, such that for any $f \in \mathcal{A}$, $\Gamma(f) \in \mathbb{M}_\infty$.

It can be seen that \mathcal{A} is an algebra (i.e. \mathcal{A} is closed w.r.t. pointwise multiplication), if it exists. In particular, by Lemma 3.4 we know \mathcal{A} is dense in $W^{1,2}$.

Theorem 3.6 (Improved Bakry-Émery theory). *Let (X, d, \mathbf{m}) be a metric measure space satisfying Assumption 2.1. Assume also the existence of an algebra \mathcal{A} in Assumption 3.5. If for any $f \in W^{1,2}$ we have the gradient estimate*

$$|DH_t f|^p \leq e^{-pKt} H_t |Df|^p, \quad \mathbf{m} - a.e. \quad (3.5)$$

for some $p \in [1, \infty)$. Then (X, d, \mathbf{m}) is a $\text{RCD}(K, \infty)$ space.

Proof. **Part 1.** Firstly, we prove

$$\Gamma(f) \Delta_{ac} \Gamma(f) - \frac{1}{4} \Gamma(\Gamma(f)) \geq \Gamma(f) \Gamma(\Delta f, f) \quad (3.6)$$

for any $f \in \mathcal{A}$.

For any $\varphi \in \text{TestF}(X, d, \mathbf{m})$, $\varphi \geq 0$ and $t > 0$, we define $F : [0, t] \mapsto \mathbb{R}$ by

$$F(s) = \int e^{-pKs} H_s \varphi \Gamma(H_{t-s} f)^{\frac{p}{2}}.$$

It can be seen that F is a C^1 function (see Lemma 2.1, [5]). From (3.5) we know $F(s) \leq F(t)$ holds for any $s \in [0, t]$. Hence $F'(s)|_{s=t} \geq 0$, which is to say

$$\begin{aligned} & \int e^{-pKs} \Delta H_s \varphi \Gamma(H_{t-s} f)^{\frac{p}{2}} - p \int e^{-pKs} H_s \varphi \Gamma(H_{t-s} f)^{\frac{p}{2}-1} \Gamma(\Delta H_{t-s} f, H_{t-s} f) \\ & \geq pK \int e^{-pKs} H_s \varphi \Gamma(H_{t-s} f)^{\frac{p}{2}} \end{aligned}$$

when $s = t$. So, letting $t \rightarrow 0$ we obtain

$$\int \Delta \varphi \Gamma(f)^{\frac{p}{2}} - p \int \varphi \Gamma(f)^{\frac{p}{2}-1} \Gamma(\Delta f, f) \geq pK \int \varphi \Gamma(f)^{\frac{p}{2}}.$$

In particular, from Lemma 2.6 and Lemma 3.2 in [15] we know $\Gamma(f)^{\frac{p}{2}} \in D(\Delta)$ and

$$\frac{1}{p}\Delta\Gamma(f)^{\frac{p}{2}} - \Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f)d\mathbf{m} \geq K\Gamma(f)^{\frac{p}{2}}d\mathbf{m}. \quad (3.7)$$

By Lemma 3.2, we know

$$\frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\Delta_{ac}\Gamma(f) + \frac{1}{2}\left(\frac{p}{2} - 1\right)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f)) \geq \Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f) + K\Gamma(f)^{\frac{p}{2}}$$

\mathbf{m} -a.e.. Then we have the inequality

$$\frac{1}{2}\Gamma(f)\Delta_{ac}\Gamma(f) + \left(\frac{p}{4} - \frac{1}{2}\right)\Gamma(\Gamma(f)) \geq \Gamma(f)\Gamma(\Delta f, f) + K\Gamma(f)^2 \quad (3.8)$$

holds \mathbf{m} -a.e..

From now, all the inequalities are considered in \mathbf{m} -a.e. sense. We denote $\frac{1}{2}\Delta_{ac}\Gamma(f) - \Gamma(\Delta f, f)$ by $\Gamma_2(f)$, and $\Delta_{ac}\Gamma(f) - \frac{1}{2}\Gamma(\Delta f, f) - K\Gamma(f)$ by $\Gamma_{2,K}(f)$, then (3.8) becomes

$$\Gamma_{2,K}(f)\Gamma(f) + \left(\frac{p}{4} - \frac{1}{2}\right)\Gamma(\Gamma(f)) \geq 0$$

\mathbf{m} -a.e.. For any real number $r \geq 0$, we say that the property $B(r)$ holds if

$$\Gamma_{2,K,r}(f) := \Gamma_{2,K}(f)\Gamma(f) + r\Gamma(\Gamma(f)) \geq 0$$

for any $f \in \text{TestF}$. In particular, (3.8) means $B(\frac{p}{4} - \frac{1}{2})$.

Now we define

$$P(r) = r - \frac{1}{4(r+1)}.$$

Then we will prove that $B(r)$ implies $B(P(r))$. We choose the smooth function $\Phi : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by

$$\Phi(\mathbf{f}) := \lambda f_1 + (f_2 - a)(f_3 - b) - ab, \quad a, b, \lambda \in \mathbb{R}.$$

Then we know

$$\begin{aligned} \Phi_{23}(\mathbf{f}) &= \Phi_{32} = a, & \Phi_{ij}(\mathbf{f}) &= 0, \quad \text{if } (i, j) \notin \{(2, 3), (3, 2)\} \\ \Phi_1(\mathbf{f}) &= \lambda, & \Phi_2(\mathbf{f}) &= f_3 - b, & \Phi_3(\mathbf{f}) &= f_2 - a. \end{aligned}$$

If $\mathbf{f} := (f, g, h) \in \mathcal{A}^3$, we know $\Phi(\mathbf{f}) \in \mathcal{A}$ by Lemma 2.4. Hence we know

$$\Gamma_{2,K}(\Phi(f))\Gamma(\Phi(f)) + r\Gamma(\Gamma(\Phi(f))) \geq 0. \quad (3.9)$$

By direct computation using Lemma 2.5 (see also Theorem 3.4, [15]), we have

$$\begin{aligned} \Gamma(\Phi(\mathbf{f})) &= g^{ij}\Phi_i\Phi_j(\mathbf{f}) \\ &= \lambda^2\Gamma(f) + (g - a)A_1 + (h - b)B_1 \end{aligned}$$

where $g^{ij} = \Gamma(f_i, f_j)$, A_1, A_2 are some additional terms.

Similarly, we have

$$\begin{aligned}
\Gamma(\Gamma(\Phi(\mathbf{f}))) &= \Gamma(g^{ij}\Phi_i\Phi_j(\mathbf{f})) \\
&= (g^{ij})^2\Gamma(\Phi_i\Phi_j) + (\Phi_i\Phi_j)^2\Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j\Gamma(g^{ij}, \Phi_i\Phi_j) \\
&= (g^{ij})^2\left[\Phi_i^2\Gamma(\Phi_j) + \Phi_j^2\Gamma(\Phi_i) + 2\Phi_i\Phi_j\Gamma(\Phi_i, \Phi_j)\right] \\
&\quad + (\Phi_i\Phi_j)^2\Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j\Gamma(g^{ij}, \Phi_i\Phi_j) \\
&= 2(g^{12})^2\lambda^2\Gamma(h) + 2(g^{13})^2\lambda^2\Gamma(g) + \lambda^4\Gamma(g^{11}) + (g-a)A_2 + (h-b)B_2 \\
&= 2\Gamma(f, g)^2\lambda^2\Gamma(h) + 2\Gamma(f, h)^2\lambda^2\Gamma(g) + \lambda^4\Gamma(\Gamma(f)) + (g-a)A_2 + (h-b)B_2.
\end{aligned}$$

We also know (see Theorem 3.4, [15] or Lemma 3.3.7, [10]) that

$$\begin{aligned}
\Gamma_2(\mathbf{f}) - K\Gamma(\Phi(\mathbf{f})) &= \lambda^2\Gamma_2(f) + 4\lambda\text{Hess}[f](g, h) + 2\left(\Gamma(g)\Gamma(h) + \Gamma(g, h)^2\right) \\
&\quad + (g-a)A_3 + (h-b)B_3 - K\lambda^2\Gamma(f).
\end{aligned}$$

Using the same argument as Theorem 3.4, [15] and Lemma 3.3.7, [10], we can replace b in the equations above by h and replace a by g . Then we obtain the following inequality from (3.9):

$$\begin{aligned}
&\lambda^2\Gamma(f) \left[\lambda^2\Gamma_2(f) + 4\lambda\text{Hess}[f](g, h) + 2\left(\Gamma(g)\Gamma(h) + \Gamma(g, h)^2\right) - K\lambda^2\Gamma(f) \right] \\
&+ r \left[2\Gamma(f, g)^2\lambda^2\Gamma(h) + 2\Gamma(f, h)^2\lambda^2\Gamma(g) + \lambda^4\Gamma(\Gamma(f)) \right] \\
&\geq 0.
\end{aligned}$$

Since $r \geq 0$ and

$$\Gamma(g)\Gamma(h) \geq \Gamma(g, h)^2,$$

we have

$$\begin{aligned}
&\Gamma(f) \left[\lambda^2\Gamma_2(f) + 4\lambda\text{Hess}[f](g, h) + 4\left(\Gamma(g)\Gamma(h)\right) - K\lambda^2\Gamma(f) \right] \\
&+ r \left[4\Gamma(f)\Gamma(g)\Gamma(h) + \lambda^2\Gamma(\Gamma(f)) \right] \\
&\geq 0.
\end{aligned}$$

Then we have

$$(\Gamma_2(f)\Gamma(f) + r\Gamma(\Gamma(f)) - K\Gamma(f)^2)\lambda^2 + 4\lambda\Gamma(f)\text{Hess}[f](g, h) + 4(r+1)\Gamma(f)\Gamma(g)\Gamma(h) \geq 0.$$

Applying Lemma 2.6 we obtain

$$(1+r)\Gamma_{2,K,r}\Gamma(f)\Gamma(g)\Gamma(h) \geq \Gamma(f)^2\text{Hess}[f](g, h).$$

Since $B(r)$ means $\Gamma_{2,K,r} \geq 0$, this inequality is equivalent to

$$(1+r)\Gamma_{2,K,r}(f)\Gamma(g)\Gamma(h) \geq \Gamma(f)\text{Hess}[f](g, h). \quad (3.10)$$

From the definition of $\text{Hess}[\cdot]$, we know

$$\text{Hess}[f](g, h) + \text{Hess}[g](f, h) = \Gamma(\Gamma(f, g), h).$$

Combining with inequality (3.10) we have

$$\begin{aligned}\sqrt{\frac{1}{1+r}}\Gamma(\Gamma(f, g), h)\sqrt{\Gamma(f)} &\leq \sqrt{\Gamma_{2,K,r}(f)\Gamma(g)\Gamma(h)} + \sqrt{\Gamma_{2,K,r}(g)\Gamma(f)\Gamma(h)} \\ &= \left(\sqrt{\Gamma_{2,K,r}(f)\Gamma(g)} + \sqrt{\Gamma_{2,K,r}(g)\Gamma(f)}\right)\sqrt{\Gamma(h)}.\end{aligned}$$

Then we fix $f, g \in \mathcal{A}$, and approximate any $h \in W^{1,2} \cap L^\infty$ with a sequence $(h_n) \subset \mathcal{A}$ converging to h strongly in $W^{1,2}$ such that

$$\Gamma(h_n) \rightarrow \Gamma(h), \quad \Gamma(h_n, \Gamma(f, g)) \rightarrow \Gamma(h, \Gamma(f, g))$$

pointwise and in $L^1(X, \mathbf{m})$. Thus we can replace h by $\Gamma(f, g)$ in the last inequality and obtain

$$\sqrt{\frac{1}{1+r}}\sqrt{\Gamma(\Gamma(f, g))\Gamma(f)} = \left(\sqrt{\Gamma_{2,K,r}(f)\Gamma(g)} + \sqrt{\Gamma_{2,K,r}(g)\Gamma(f)}\right). \quad (3.11)$$

Let $f = g$ in (3.11) we have

$$\frac{1}{1+r}\Gamma(\Gamma(f))\Gamma(f) \leq 4\Gamma_{2,K,r}(f)\Gamma(f).$$

Therefore,

$$\left(\frac{1}{4} \frac{1}{1+r} - r\right)\Gamma(\Gamma(f))\Gamma(f) \leq \Gamma_{2,K}(f)\Gamma(f).$$

In other words, we have $B(P(r))$.

For any $a_0 \geq 0$, we define the sequence by the recursive formula

$$a_{n+1} = P(a_n).$$

From Lemma 3.3 we know there exists $a_0 \geq \frac{p}{4} - \frac{1}{2}$ and $N_0 \in \mathbb{N}$ such that $a_{N_0} = 0$. Then we know $B(a_0)$ from (3.8). By induction and the result above, we can see that $B(a_{N_0})$ is true. In particular, we know $B(\frac{1}{4})$ holds.

Part 2. In the first part, we know the inequality (3.7) is equivalent to $B(\frac{p}{4} - \frac{1}{2})$ for any $p > 2$. Therefore, from (3.6) we know

$$\frac{1}{p_n}\Delta\Gamma(f)^{\frac{p_n}{2}} - \Gamma(f)^{\frac{p_n}{2}-1}\Gamma(\Delta f, f)d\mathbf{m} \geq K\Gamma(f)^{\frac{p_n}{2}}d\mathbf{m}.$$

for any $p_n = 2 + \frac{1}{2^n}$, where $n \in \mathbb{N}$.

First of all, we assume that $f \in \mathcal{A}$. Let $\varphi \in \text{TestF}$, and $t > 0$, we define $F : [0, t] \mapsto \mathbb{R}$ by

$$F(s) = \int e^{-p_n K s} H_s \varphi \Gamma(H_{t-s} f)^{\frac{p_n}{2}}.$$

We know:

$$\begin{aligned}F'(s) &= \int e^{-p_n K s} \Delta H_s \varphi \Gamma(H_{t-s} f)^{\frac{p_n}{2}} - p_n \int e^{-p_n K s} H_s \varphi \Gamma(H_{t-s} f)^{\frac{p_n}{2}-1} \Gamma(\Delta H_{t-s} f, H_{t-s} f) \\ &\quad - p_n K \int e^{-p_n K s} H_s \varphi \Gamma(H_{t-s} f)^{\frac{p_n}{2}} \geq 0\end{aligned}$$

for any $s \in [0, t]$. Hence $F(t) \geq F(0)$, i.e.

$$\int \varphi e^{-p_n K t} H_t \Gamma(f)^{\frac{p_n}{2}} \geq \int \varphi \Gamma(H_t f)^{\frac{p_n}{2}}.$$

Since φ is arbitrary, by Lemma 3.4 we know $\Gamma(H_t f)^{\frac{p_n}{2}} \leq e^{-p_n K t} H_t \Gamma(f)^{\frac{p_n}{2}}$ \mathbf{m} -a.e.. Therefore, for \mathbf{m} -a.e. $x \in X$ we know

$$\Gamma(H_t f)^{1+\frac{1}{2n+1}} \leq e^{-2+\frac{1}{2n} K t} H_t \Gamma(f)^{1+\frac{1}{2n+1}}.$$

Letting $n \rightarrow \infty$, by dominated convergence theorem we know

$$\Gamma(H_t f) \leq e^{-2Kt} H_t \Gamma(f) \quad \mathbf{m} - \text{a.e.} \quad (3.12)$$

for all $f \in \mathcal{A}$. At last, combining the density of \mathcal{A} in TestF and Lemma 3.4, by lower semi-continuity (see preliminary section), we know (3.12) holds for all $f \in W^{1,2}$.

Then, by Theorem 4.17 in [5] we know (X, d, \mathbf{m}) is a $\text{RCD}(K, \infty)$ space. \square

As an applications, we have the following theorems concerning the self-improvement of gradient estimate.

Theorem 3.7 (Self-improvement of gradient estimate). *Let $M := (X, d, \mathbf{m})$ be a $\text{RCD}(K, \infty)$ metric measure space. Assume that X can be covered by (convex) open sets $\{\Omega_i\}_i$ such that $(\Omega_i, d, \mathbf{m})$ is $\text{RCD}(K_i, \infty)$ for some $K_i \in \mathbb{R}$, $i \in \mathbb{N}^+$. If for any $f \in W^{1,2}$ we have the gradient estimate*

$$|DH_t f|^p \leq e^{-pKt} H_t |Df|^p, \quad \mathbf{m} - \text{a.e.} \quad (3.13)$$

for some $p \in [1, \infty)$. Then (X, d, \mathbf{m}) is a $\text{RCD}(K, \infty)$ space.

Proof. Since $(\Omega_i, d, \mathbf{m})$ is $\text{RCD}(K_i, \infty)$, by Lemma 3.2 in [15] we know that $\text{TestF}(\Omega_i)$ itself is an algebra in Assumption 3.5. From the definition of measure valued Laplacian and the locality of weak gradient, we know $\text{TestF}(M)|_{\Omega_i} \subset \text{TestF}(\Omega_i)$ where $\text{TestF}(M)|_{\Omega_i} := \{f|_{\Omega_i} : f \in \text{TestF}(M)\}$. Now we assert that $\text{TestF}(M)|_{\Omega_i}$ is a dense subset in $\text{TestF}(\Omega_i)$ with respect to the $W^{1,2}(\Omega_i)$ norm.

Let $f \in \text{TestF}(\Omega_i)$, we know there exists a sequence of cut-off functions $(\chi_n) \subset \text{TestF}(\Omega_i)$ which are supported on Ω_i , such that $f\chi_n \rightarrow f$ in $W^{1,2}(\Omega_i)$ (see Lemma 6.7 in [6] for the construction of such cut-off functions). Since $\text{TestF}(\Omega_i)$ is an algebra, we know $f\chi_n \in \text{TestF}(\Omega_i)$. To prove the assertion, we just need to prove that there exists $g \in \text{TestF}(M)$ such that $g|_{\Omega_i} = f\chi_n$. In fact, we define

$$g(x) = \begin{cases} f\chi_n, & x \in \Omega_i \\ 0, & x \in \Omega_i^c, \end{cases}$$

It can be checked that $g \in \text{TestF}(M)$ from the definition, by noticing the locality of weak gradient and uniqueness of the measure valued Laplacian. Applying Theorem 3.6 we know (X, d, \mathbf{m}) is $\text{RCD}(K', \infty)$.

Then we know there exists a dense sub-algebra of $\text{TestF}(\Omega_i)$ whose elements satisfy the inequality (3.8) on Ω_i . Applying Theorem 3.6 to the space $(\Omega_i, d, \mathbf{m})$, we know $(\Omega_i, d, \mathbf{m})$ is $\text{RCD}(K, \infty)$ for any i . At last, by local-to-global property (see e.g. [16]) we know (X, d, \mathbf{m}) is $\text{RCD}(K, \infty)$. \square

As a direct consequence of Theorem 3.7, we prove the following well-known proposition.

Corollary 3.8 (See e.g. [14] and [17]). *Let $(M, g, e^{-V}\text{Vol})$ be a weighted Riemannian manifold where $V \in C^\infty(M)$. We denote the semi-group whose generator is $\Delta^V := \Delta - \langle \nabla V, \nabla \cdot \rangle$ by H_t . For any $K \in \mathbb{R}$ the following properties are equivalent*

$$1) \text{ Ricci}^V := \text{Ricci}_M + \text{Hess}V \geq K,$$

$$2) \text{ for all } f \in C_c^\infty(M), \text{ all } x \in M \text{ and } t \geq 0$$

$$|\text{DH}_t f|^p(x) \leq e^{-pKt} H_t |Df|^p(x) \quad (3.14)$$

for some $p \in [1, \infty)$.

$$3) \text{ for all } f \in C_c^\infty(M), \text{ all } x \in M \text{ and } t \geq 0$$

$$|\text{DH}_t f| \leq e^{-Kt} H_t |Df|. \quad (3.15)$$

Proof. It is known that $(M, g, e^{-V}\text{Vol})$ is $\text{RCD}(K, \infty)$ if and only if the synthetic Ricci tensor $\text{Ricci}^V := \text{Ricci}_M + \text{Hess}V \geq K$. Since for every point on M we can find a bounded convex neighbourhood of it. By Theorem 3.7 we know the assertion is true. An alternative proof is to use Theorem 3.6 by noticing that $C_c^\infty(M)$, the space of smooth functions with bounded support is a good algebra in Assumption 3.5. \square

At the end, we have the following Proposition which tells us that p -gradient estimate can characterize the synthetic Ricci bound on $\text{RCD}(K, \infty)$ spaces. See [10] (and [12]) for another approach to characterize Ricci bound using measure valued Ricci tensors.

Proposition 3.9 (Self-improvement of gradient estimate). *Let (X, d, \mathbf{m}) be a $\text{RCD}(K, \infty)$ metric measure space. If for any $f \in W^{1,2}$ we have the gradient estimate*

$$|\text{DH}_t f|^p \leq e^{-pK't} H_t |Df|^p, \quad \mathbf{m} - a.e. \quad (3.16)$$

for some $p \in [1, \infty)$. Then (X, d, \mathbf{m}) is a $\text{RCD}(K', \infty)$ space. In particular, we know

$$|\text{DH}_t f| \leq e^{-K't} H_t |Df|, \quad \mathbf{m} - a.e.. \quad (3.17)$$

Proof. It is proved in Lemma 3.2, [15] that $f \in \mathbb{M}_\infty$ for any $f \in \text{TestF}$. In particular, TestF itself is an algebra in Assumption 3.5. Applying Theorem 3.6 we know (X, d, \mathbf{m}) is $\text{RCD}(K', \infty)$. From Corollary 4.3, [15] we obtain the improved gradient estimate (3.17). \square

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